

From solid to granular gases: the steady state for sands

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Abstract

This paper aims at extending the well known critical state concept, associated with quasi-static conditions, by accounting for the role played by the strain rate when focusing on the steady, simple shear flow of a dry assembly of identical, elastic spheres. A constitutive model is proposed and interpreted in the framework of visco-plasticity. The model includes the granular temperature, a measure of the degree of agitation of the particles, as an additional state variable. The stresses of the system are associated with enduring, frictional contacts among particles involved in force chains and nearly instantaneous collisions. When the first mechanism prevails, the material behaves like a solid, and constitutive models of soil mechanics hold; whereas when inelastic collisions dominate, the material flows like a granular gas and kinetic theories apply. The predictions of the model for the steady, simple shear flow of 1 mm sand are discussed. At large values of the normal stress and small values of the shear rate, the theory predicts a strain rate softening that can have important implications e.g. on the evolution of landslides.

1 Introduction

Since the pioneering works of Roscoe et al. (1958) and Schofield and Wroth (1968), the critical state concept for granular materials, like sands and gravels, has been introduced to describe the mechanical behavior of granular materials under evolving quasi-static conditions. Experimental (especially using strain controlled triaxial tests, CITATION) and theoretical investigations (Been and Jefferies, 1986; Gajo and Muir Wood, 1999) have been performed, also accounting for peculiar mechanical processes like grain crushing (Pestana and Whittle, 1995) and grain segregation (CITATION). However, the time factor and the role of the strain rate have received much less attention by the Geotechnical community (di Prisco and Imposimato, 1996; Benedetto and Tatsuoka, 1997; di Prisco et al., 2000).

On the contrary, the most of works published within the granular flow

community (e.g., see Goldhirsch, 2003) deals with the rheology of granular materials at large strain rates and low to moderate concentration, far from the quasi-static conditions, and determined using simple shear tests (rheometers) under steady conditions. Within the framework of continuum mechanics, the granular temperature, defined as the mean square of the velocity fluctuations, quantitatively describes the degree of agitation of the system. According to kinetic theories (Jenkins and Savage, 1983; Savage, 1984; Goldhirsch, 2003), the inelastic collisions associated with the random motion of the grains represent the main mechanism to dissipate the energy of the system. From the Geotechnical viewpoint, under those conditions (granular gaseous or collisional state), the force chains within the medium forming the granular skeleton disappear.

There are several practical problems where the granular material encompasses a transition from a solid to a more gaseous state, suggesting therefore that a collaboration between the two above mentioned communities would be fruitful. For instance, the landslide risk evaluation requires to model both the inception and the evolution of the gravitational collapse. The increasing success of computational tools in handling large deformations (CITATION) suggests that such an ambitious goal is now possible. This stimulates the need for constitutive models of the mechanical response of granular materials under both quasi-static and collisional conditions. A first step is the extension of the critical state concept, interpreted hereafter as a sort of limit condition for the steady state at vanishingly small strain rate, by employing the granular temperature as in di Prisco and Pisanò (2008), as an additional state variable for the system. When the granular temperature, T , is large, the stored energy of the system is prevalently kinetic; whereas, at small T , the stored energy of the system is mainly elastic. In this perspective, a recent constitutive model (Berzi et al., 2011) for the granular material, valid in both quasi-static and collisional conditions, is slightly modified, mechanical interpreted in the light of visco-plasticity and parametrically discussed. This paper wants to suggest a road map, allowing the Geotechnicians to get outside from their “one-dimensional world” and discovering, as for the inhabitants of Flatland in the well-known masterpiece of Abbott (1884), a “marvelous” multi-dimensional state-variable universe.

2 Theory

The Geotechnical community usually associates the concept of critical state with a non-evolving state reached after a progressive increase of strain, at a vanishingly small strain-rate. At the critical state, an ideal mechanism of yielding is assumed to develop within the specimen: the external work is totally dissipated by frictional processes at the contact level (disregarding

both crushing and damage); the micro-structure does not evolve and, consequently, the void index, e , remains constant (i.e., segregation is inhibited). Focusing on triaxial tests (recently also by means of distinct element numerical simulations, Thornton and Zhang, 2003), Geotechnicians have traditionally defined the locus of the critical state using the void index, the Terzaghi (CITATION) effective pressure and the deviator (defined as the difference between the axial and radial stress). Conversely, only few works have dealt with simple shear flows (Fig. 1), which instead represents the most studied configuration in determining the rheology of granular gases (Bagnold, 1954; GDR-MiDi, 2004; da Cruz et al., 2005; Mitarai and Nakanishi, 2007).

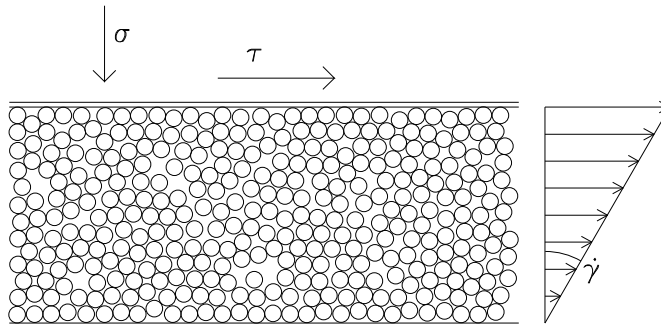


Figure 1: Simple shear flow configuration.

Limiting the analysis, for sake of simplicity, on the homogeneous simple shear of an assembly of identical, dry spheres of diameter d , the variables governing the problem are the shear stress τ , the normal stress in the transversal direction, σ , the void index, or alternatively, the concentration $\nu = 1/(1 + e)$, the shear strain, the strain rate $\dot{\gamma}$ and the granular temperature T . In the realm of Geotechnique, the strain rate is usually taken to be zero, and therefore disregarded, and the granular temperature is ignored. In the realm of granular flows, the shear strain is infinite, as for classic fluids, therefore not influencing the problem. In our view, the two realms are strictly connected, and the critical state, for which the shear strain is infinite and both the strain rate and the granular temperature are zero, represents the boundary between them.

Several authors (Savage, 1998; Johnson and Jackson, 1987, 1990; Lee and Huang, 2010; Berzi et al., 2011) have suggested to model the stresses in granular materials as the linear combination of two contributions:

$$\begin{aligned}\sigma &= \sigma_q + \sigma_c, \\ \tau &= \tau_q + \tau_c.\end{aligned}\tag{1}$$

Here and in what follows, the subscript q (quasi-static) and c (collisional) refer to quantities associated with enduring, frictional contacts of particles

involved in force chains (soil skeleton) and nearly instantaneous collisions, respectively. In steady, simple shear flows in absence of gravity, both the shear and the normal stress are constant in the flow. Also the granular temperature is constant, so that the flux of fluctuating energy of the particles is zero (Mitarai and Nakanishi, 2007). Hence, the balance of fluctuating energy reduces to (Jenkins, 2007; Berzi et al., 2011)

$$\tau_c \dot{\gamma} = \Gamma, \quad (2)$$

where the term on the left hand side represents the production of fluctuating energy and Γ is the rate of energy dissipation in collisions. Unlike suggested in other works (Lee and Huang, 2010), enduring contacts among particles in force chains cannot produce fluctuating energy so that only the collisional component of the shear stress is present in Eq. (2).

2.1 Quasi-static contribution

The quasi-static component of the shear stress is assumed to be proportional to the quasi-static component of the normal stress through the tangent of the critical friction angle ϕ_c , (Schofield and Wroth, 1968; Muir Wood, 2004):

$$\tau_q = \sigma_q \tan \phi_c. \quad (3)$$

The friction angle ϕ_c is a function of both the inter-particle friction coefficient μ and the simple shear constraints (di Prisco and Pisanò, 2011).

For dimensional reasons, the constitutive relation for the normal stress reads

$$\sigma_q = f_0 \frac{K}{d} \quad (4)$$

where the particle stiffness K is equal to $\pi d E / 8$ in the case of linear elastic contacts (Ji and Shen, 2008), with E the Young's modulus, and f_0 is solely a function of the concentration.

The function f_0 must vanish when the concentration is equal to the random loose packing value, ν_{rlp} , the minimum concentration at which a disordered packing exists (Song et al., 2008). At the random loose packing, the quasi-static normal stress is zero, and the granular material undergoes a phase transition to a purely collisional regime. The concentration at random loose packing is a decreasing function of the inter-particle friction coefficient μ (Song et al., 2008; Silbert, 2010). For frictionless particles, i.e., $\mu = 0$, ν_{rlp} coincides with the random close packing, $\nu_{rcp} = 0.636$, the densest possible disordered packing of identical spheres (Song et al., 2008). On the other hand, f_0 must diverge at another critical value, ν_s , at which the force chains span the entire domain and a shear rigidity develops (Jenkins and Berzi, 2010). Therefore, we take

$$f_0 = \max \left(a \frac{\nu - \nu_{rlp}}{\nu_s - \nu}, 0 \right), \quad (5)$$

where a is a dimensionless material coefficient (Berzi et al., 2011). Experimental investigations on the critical state of identical spheres are though rare. To our knowledge, only Wroth (1958) performed experiments on the critical state of 1 mm stainless steel spheres ($K = 8.25 \cdot 10^7$ Pa m) using a shear cell (Muir Wood, 2007). The experiments confirm that the ratio of τ_q to σ_q is constant and that f_0 is a unique function of the concentration. The solid line in Fig. 2(a) represents the theoretical expression of Eq. (5), with $\nu_s = 0.619$, $\nu_{rlp} = 0.598$ and $a = 1.8 \cdot 10^{-6}$, obtained from linear regression. The data of Fig. 2(a) are plotted in terms of f_0 against void index in Fig. 2(b).

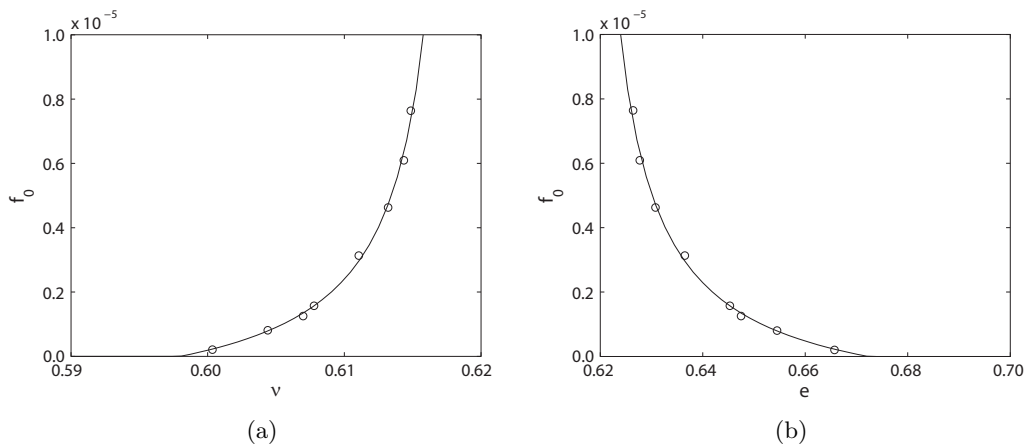


Figure 2: Experimental (cricles, after Wroth, 1958) and theoretical (solid line) coefficient f_0 for steel spheres as function of (a) concentration and (b) void index.

2.2 Collisional contribution

The constitutive relations for the collisional stresses and the rate of dissipation of fluctuating energy are those proposed by Garz3 and Dufty (1999), as modified by Jenkins and Berzi (2010),

$$\sigma_c = \rho_p f_1 f_4 T, \quad (6)$$

$$\tau_c = \rho_p d f_2 f_4 T^{1/2} \dot{\gamma}, \quad (7)$$

and

$$\Gamma = \rho_p \frac{f_3}{L} f_4 T^{3/2}. \quad (8)$$

The coefficients f_1 , f_2 and f_3 are reported in Tab. 1. There, ϵ is taken to be an effective coefficient of restitution that depends on the normal coefficient of restitution (ratio of pre to post collisional relative velocity between

$f_1 = 4\nu GF$
$f_2 = \frac{8J}{5\pi^{1/2}}\nu G$
$f_3 = \frac{12}{\pi^{1/2}}(1 - \epsilon^2)\nu G$
$G = \nu g_0$
$g_0 = \begin{cases} \frac{(2 - \nu)}{2(1 - \nu)^3} & \nu \leq 0.49 \\ \frac{5.69(\nu_s - 0.49)}{\nu_s - \nu} & \nu > 0.49 \end{cases}$
$F = \frac{1 + \epsilon}{2} + \frac{1}{4G}$
$J = \frac{1 + \epsilon}{2} + \frac{\pi}{32} \frac{[5 + 2(1 + \epsilon)(3\epsilon - 1)G][5 + 4(1 + \epsilon)G]}{[24 - 6(1 - \epsilon)^2 - 5(1 - \epsilon^2)]G^2}$
$\frac{L}{d} = \max \left[1, \left(\frac{1}{2} c \frac{G^{1/3}}{T^{1/2}} d\dot{\gamma} \right) \right]$

Table 1: List of expressions for the collisional contribution to the stresses.

colliding particles in the normal impact direction), the tangential coefficient of restitution in a sticking collision, and the Coulomb friction coefficient that characterizes sliding collisions (Jenkins and Zhang, 2002). In Tab. 1, g_0 is the radial distribution function. For concentration higher than 49%, we adopt the expression of g_0 suggested by Torquato (1995), and identify the singularity as ν_s . In the elastic limit, i.e., when $\epsilon = 1$, ν_s equals ν_{rcp} (Torquato, 1995). In Eq. (8), L is the correlation length, accounting for the decrease in the collisional energy dissipation due to the correlated motion of particles that is likely to occur when the flow is dense (Jenkins, 2006, 2007; Jenkins and Berzi, 2010). In its expression, reported in Tab. 1, c is a dimensionless material coefficient of order unity.

The coefficient f_4 in Eqs. (6)-(8), not present in the constitutive relations of Jenkins and Berzi (2010), takes into account the influence of the particle stiffness on the collisions. Following Hwang and Hutter (1995),

$$f_4 = \left[1 + 2 \frac{d}{s} \left(\frac{\rho_p T}{E} \right)^{1/2} \right]^{-1}. \quad (9)$$

where s is the mean separation distance among particles. At equilibrium,

the latter can be identified as the mean free path (mean distance traveled by a particle between two successive collisions), i.e., as the product of the mean fluctuating velocity and the mean collision interval. Hence, in the context of classic kinetic theories (Chapman and Cowling, 1970),

$$s = \frac{\sqrt{2} d}{12 G}. \quad (10)$$

Using Eqs. (1), (4) and (6),

$$T = \frac{\sigma - f_0 K/d}{\rho_p f_1 f_4}. \quad (11)$$

With this and Eq. (9),

$$\frac{1}{f_4} - 2 \frac{d}{s} \sqrt{\frac{\pi}{8 f_1} \left(\frac{\sigma d}{K} - f_0 \right)} \frac{1}{\sqrt{f_4}} - 1 = 0, \quad (12)$$

that gives

$$f_4 = \frac{2}{2 + \mathcal{A} + \sqrt{\mathcal{A}^2 + 4\mathcal{A}}}, \quad (13)$$

where

$$\mathcal{A} = \frac{36\pi G^2}{f_1} \left(\frac{\sigma d}{K} - f_0 \right). \quad (14)$$

As expected, f_4 tends to one as K tends to infinity.

2.3 Constitutive model

By substituting Eqs. (7) and (8) into (2), and using the constitutive expression for L of Tab. 1, the granular temperature results an algebraic function of the shear rate,

$$T = d^2 f_5 \dot{\gamma}^2 \quad (15)$$

with

$$f_5 = \frac{L f_2}{d f_3}, \quad (16)$$

and

$$\frac{L}{d} = \max \left[1, \left(\frac{c^2 G^{2/3} f_3}{4 f_2} \right)^{1/3} \right]. \quad (17)$$

Using Eq. (15) into Eqs. (6) and (7), the expressions for the total stresses in steady, simple shear flows read

$$\left\{ \begin{array}{l} \sigma = \frac{K}{d} f_0 + \rho_p d^2 f_1 f_4 f_5 \dot{\gamma}^2 \\ \tau = \frac{K}{d} f_0 \tan \phi_c + \rho_p d^2 f_2 f_4 f_5^{1/2} \dot{\gamma}^2, \end{array} \right. \quad (18a)$$

$$(18b)$$

Eqs. (18) can be written as

$$\begin{cases} 1 - \frac{K}{\sigma d} f_0 - \frac{\gamma_1^2}{\gamma_2^2} \left(\frac{\tau}{\sigma} - \tan \phi_c \right) = 0 & (19a) \\ \frac{\tau}{\sigma} - \tan \phi_c - \frac{t_m^2}{\gamma_1^2} \dot{\gamma}^2 = 0 & (19b) \end{cases}$$

where $t_m = d(\rho_p \nu / \sigma)^{1/2}$ is the microscopic time scale associated with the rearrangement of the particles (GDR-MiDi, 2004), and

$$\begin{aligned} \gamma_1 &= \left[\frac{\nu}{f_4 (f_2 f_5^{1/2} - \tan \phi_c f_1 f_5)} \right]^{1/2}, \\ \gamma_2 &= \left[\frac{\nu}{f_1 f_4 f_5} \right]^{1/2}. \end{aligned} \quad (20)$$

Eq. (19b) provides

$$\dot{\gamma} = \frac{\gamma_1}{t_m} \left(\frac{\tau}{\sigma} - \tan \phi_c \right)^{1/2}, \quad (21)$$

that can be interpreted, in the visco-plastic framework, as

$$\dot{\gamma} = \tilde{\gamma} \Phi(\mathcal{F}), \quad (22)$$

where $\Phi(\mathcal{F})$ is the viscous nucleus function of the yield locus \mathcal{F} (Perzyna, 1966). In this case,

$$\Phi(\mathcal{F}) = (\mathcal{F})^{1/2}, \quad (23)$$

and

$$\mathcal{F} = \frac{\tau}{\sigma} - \tan \phi_c. \quad (24)$$

In Eq. 22, $\tilde{\gamma}$ is the fluidity parameter,

$$\tilde{\gamma} = \frac{\gamma_1}{t_m}, \quad (25)$$

that is not constant, unlike assumed in previous works (CITARE!!).

3 Discussion and results

The condition that the normal stresses of granular material must always be positive implies, from (18a),

$$1 - \frac{K}{d\sigma} f_0 \geq 0. \quad (26)$$

Given the value of σ , this condition reads

$$\nu \leq \nu_m \quad (27)$$

where

$$\nu_m = \frac{a\nu_{rlp}}{a + \sigma d/K} + \frac{\nu_s}{1 + a(\sigma d/K)^{-1}}. \quad (28)$$

For large values of $\sigma d/K$, ν_m approaches ν_s ; on the other hand, ν_m tends to ν_{rlp} when $\sigma d/K$ is small.

Fig. 3 shows the qualitative phase diagram on the plane $\mu - \nu$ for the steady, simple shear flow of inelastic spheres, i.e., $\epsilon < 1$ and $\nu_s < \nu_{rcp}$.

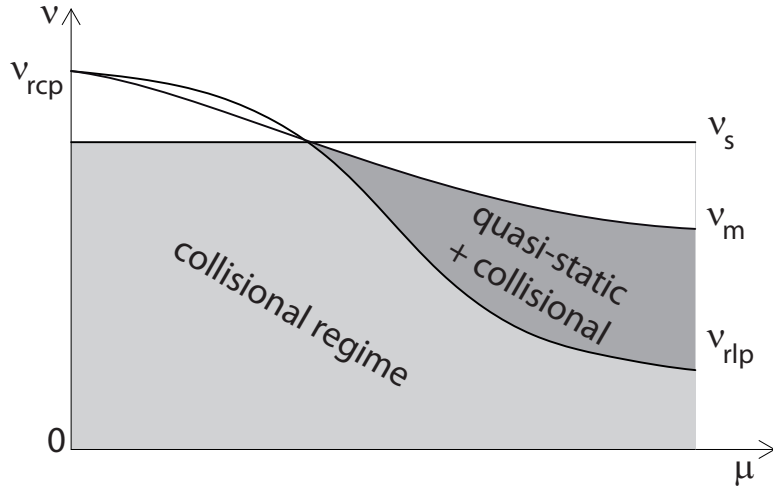


Figure 3: Phase diagram for the steady, simple shear flow of inelastic spheres.

For a given values of the inter-particle friction μ , the concentration decreases as the shear rate increases (the maximum value is when $\dot{\gamma} = 0$, i.e., at the critical state, when the collisional stresses vanish). For small values of μ , ν_{rlp} is greater than ν_s (here assumed to be constant, in absence of clear evidences of its possible dependence on μ), so that the quasi-static stresses are zero: the maximum concentration therefore coincides with ν_s and the steady, simple shear flow is always in the collisional regime. At larger μ , ν_{rlp} is lower than ν_s : the concentration at the critical state is ν_m , and quasi-static and collisional stresses coexist in the range between $\nu_{rlp} \leq \nu \leq \nu_m$. At the value of $\dot{\gamma}$ which corresponds to a concentration equal to ν_{rlp} , the quasi-static stresses vanish and the material undergoes a phase transition to the collisional regime. The range of coexistence of quasi-static and collisional stresses depends on the ratio $\sigma d/K$, which affects the value of ν_m . In particular, for small values of $\sigma d/K$, i.e., small values of the total normal stress or large values of the particle stiffness, ν_m approaches ν_s , as already mentioned, thus reducing the range of influence of the quasi-static stresses

on the flow.

Physical and numerical experiments on steady, simple shear flows can be performed (i) by imposing the normal stress, and measuring the concentration (or alternatively the void index) and the shear stress as functions of the shear rate (pressure-imposed); (ii) by imposing the concentration (void index), and measuring the normal and shear stress as functions of the shear rate (concentration-imposed). The results of the two configurations are equivalent, if dimensionless quantities are employed (da Cruz et al., 2005). Nonetheless, here we will apply our model to pressure-imposed flows, more common in the Geotechnical community.

The model parameters which affect the constitutive relations (19) can be subdivided into (i) micro-mechanical parameters, characteristics of the single particle (i.e., ρ_p , d , K , μ and ϵ); (ii) macro-mechanical parameters, characteristics of the “continuum” medium (i.e., ν_{rlp} , ν_s , $\tan \phi_c$, a and c). As previously mentioned, micro and macro-mechanical parameters are related to each other: the inter-particle friction coefficient affects the concentration at random loose packing and the critical friction angle, and the coefficient of collisional restitution influences the concentration at which the shear rigidity develops. Also, ϵ and μ are not, in principle, totally independent.

Here, the predictions of the model for sand ($d = 1$ mm) are shown (Fig. 4). Unfortunately, there are not enough data on steady, simple shear flows of sand to infer the values of all the above mentioned parameters. Whenever necessary, we make use of values appropriated for other types of granular material, assumed sufficiently close to those for sand. Hence, we take: $\rho_p = 2600$ kg/m³; $K = 2.8 \cdot 10^7$ Pa m, from the Young’s modulus for quartz; $\epsilon = 0.6$ and $c = 0.5$, appropriated for glass spheres (Jenkins and Berzi, 2010); $a = 1.8 \times 10^{-6}$, and $\nu_s = 0.619$, from Wroth’s experiments on stainless steel spheres (see section 2.1); $\tan \phi_c = 0.5$, the tangent of the angle of repose obtained by Forterre and Pouliquen (2003) for 0.8 mm sand; $\nu_{rlp} = 0.55$ as appropriated for very frictional particles (Silbert, 2010).

Fig. 4a shows that, at the lowest values of the applied normal stress, the stress ratio τ/σ increases with the shear rate when the void index is less than one, and decreases when e exceeds unity (purely collisional regime). We refer to the latter condition as a “strain rate softening” (di Prisco et al., 2000), although the reduction in the stress ratio for increasing values of $\dot{\gamma}$ is not associated with an evolutionary process, but with a succession of steady states. This strain rate softening in the purely collisional regime has been confirmed by numerical simulations on unbounded shear flows (Mitarai and Nakanishi, 2007). On the other hand, when the applied normal stress is sufficiently large, the present theory predicts an additional strain rate softening at values of e less than one, when both collisional and quasi-static stresses coexist (Fig. 4a).

Fig. 5 shows the results of the present theory when the quasi-static stresses are ignored. This permits to emphasize some key predictions of the theory

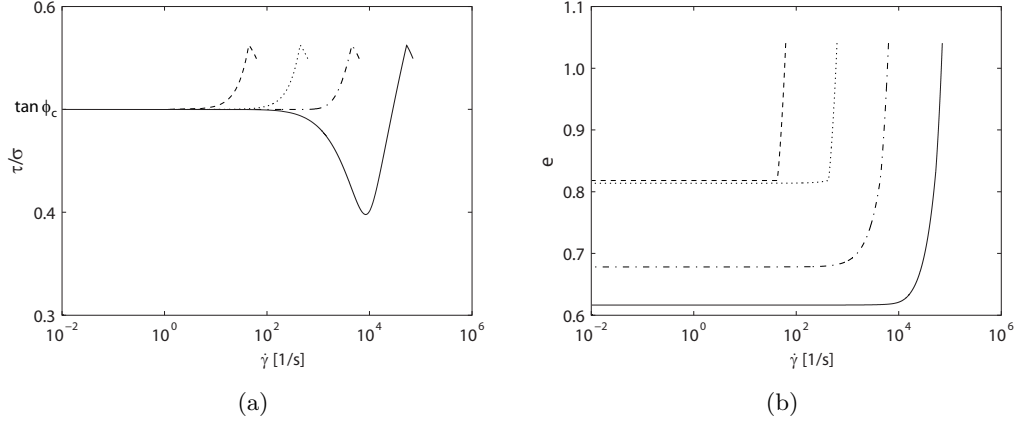


Figure 4: Theoretical (a) stress ratio and (b) void index as functions of the shear rate for 1 mm sand, at different values of the applied normal stress: $\sigma = 10$ Pa, dashed lines; $\sigma = 10^3$ Pa, dotted lines; $\sigma = 10^5$ Pa, dash-dotted lines; $\sigma = 10^7$ Pa, solid lines.

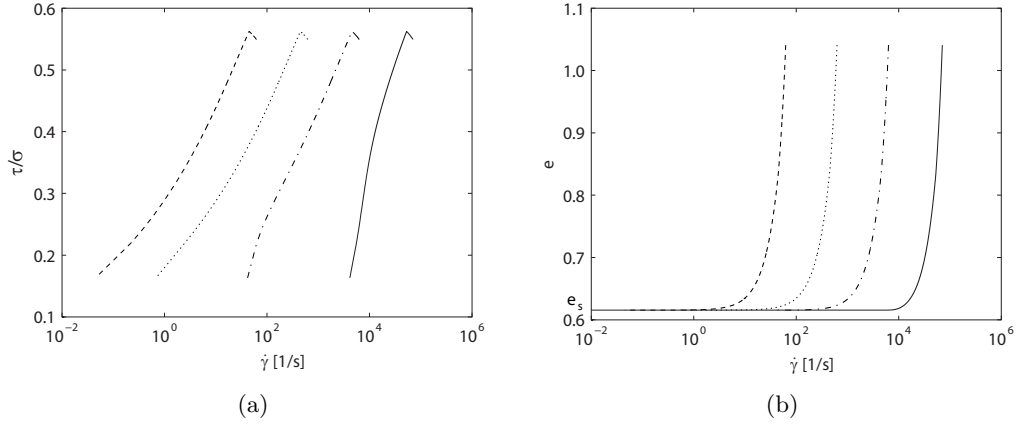


Figure 5: Same as in Fig. 4, but for the fact that the quasi-static stresses are ignored (purely collisional model).

that can be tested in numerical simulations. First, the value of the void index for $\dot{\gamma} \rightarrow 0$ would be independent on σ in a purely collisional model. Second, a purely collisional model cannot predict the asymptotic approach of the stress ratio to the critical friction angle for $\dot{\gamma} \rightarrow 0$ (da Cruz et al., 2005).

The condition for the presence of the strain rate softening in the regime when both quasi-static and collisional stresses coexist (Fig. 4a) can be de-

rived from Eq. (21). Indeed, for the shear rate being a real number,

$$\frac{\tau/\sigma - \tan \phi_c}{f_4 \left(f_2 f_5^{1/2} - \tan \phi_c f_1 f_5 \right)} > 0. \quad (29)$$

Hence, the strain rate softening ($\tau/\sigma < \tan \phi_c$) can occur if

$$f_2 f_5^{1/2} - f_1 f_5 \tan \phi_c < 0, \quad (30)$$

given that f_4 is always positive. By using the expressions of Tab. 1, Eq. 30 gives

$$\nu > \nu^*, \quad (31)$$

where

$$\nu^* = \nu_s \frac{\mathcal{B}^9 (\tan \phi_c)^{-9}}{\mathcal{B}^9 (\tan \phi_c)^{-9} + 5.69 (\nu_s - 0.49)}, \quad (32)$$

with

$$\mathcal{B} = \left[\frac{48}{5\pi(1+\epsilon)^2} \right]^{1/2} \left[\frac{(1-\epsilon^2)^2 J^4}{15c^2} \right]^{1/6}, \quad (33)$$

where J is calculated from Tab. 1 in the dense limit, i.e., for $G \rightarrow \infty$ (Jenkins and Berzi, 2010). If ν_m is larger than ν^* , there is a “strain rate softening” in the range of concentration between ν^* and ν_m . By using Eq. 28, this corresponds to

$$\sigma > a \frac{(\nu^* - \nu_{rlp}) K}{(\nu_s - \nu^*) d}. \quad (34)$$

Fig. 6 illustrates the dependence of ν_m on σ ; there, the gray area represents the fulfillment of conditions (31) and (34).

For $\nu > \nu^*$, the visco-plastic interpretation of the model (Eq. 21) still holds, if one allows the fluidity parameter to be an imaginary number. The dependence of the fluidity parameter on the void index for different values of σ is depicted in Fig. 7. The gray area represents the range of void index for which $\tilde{\gamma}$ is imaginary.

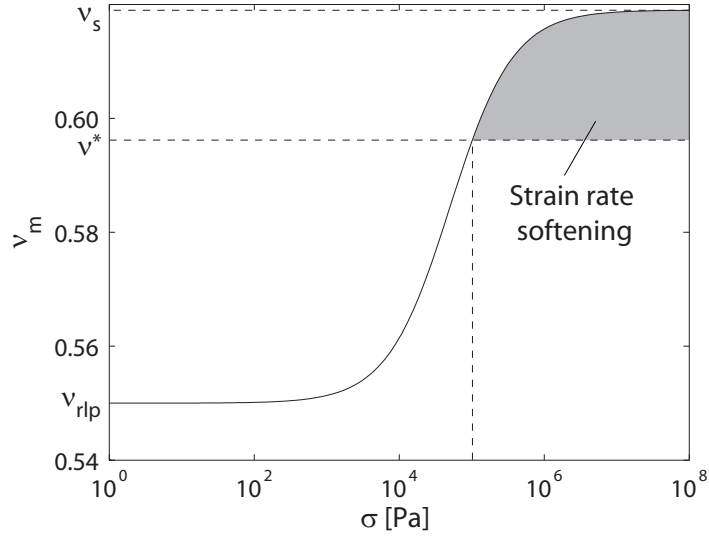


Figure 6: Maximum concentration attained in the steady, simple shear flow of 1 mm sand as function of the applied normal stress. The gray area represents the range of existence of the strain rate softening when both collisional and quasi-static stresses coexist.

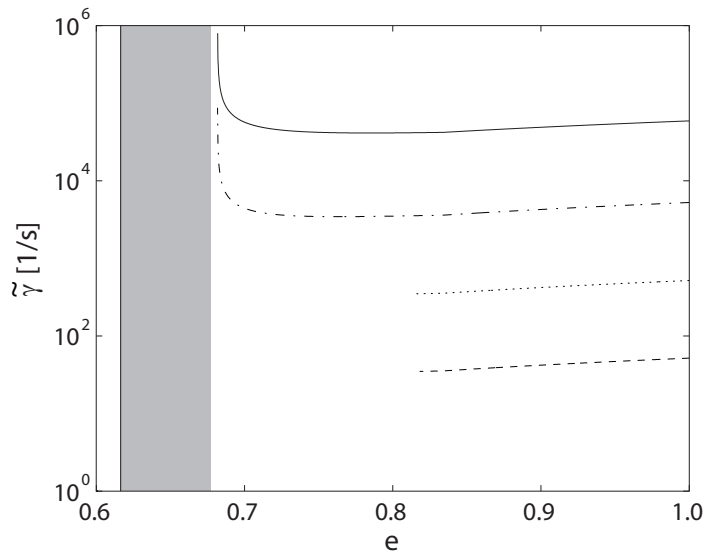


Figure 7: Fluidity parameter as function of the void index for $\sigma = 10$ Pa (dashed line), $\sigma = 10^3$ Pa (dotted line), $\sigma = 10^5$ Pa (dash-dotted line), $\sigma = 10^7$ Pa (solid line).

4 Concluding remarks

This work has provided a theoretical framework, in which both standard Geotechnical constitutive models, based on the critical state theory, and kinetic theories of granular gases can be reconciled. In particular, the steady state condition of a granular material (sand) under simple shear has been analyzed using a constitutive model recently proposed by the authors, where both enduring contacts among particles involved in force chains and nearly instantaneous collisions are considered. The interpretation of the constitutive model in the light of standard visco-plasticity is a first step towards an evolving constitutive model capable of describing the mechanical behaviour of granular material under both solid-like and fluid-like conditions. Taking into account the stiffness of the particles permits to highlight the role of the applied normal stress on the dependence between the stress ratio and the void index. Indeed, for large values of the normal stress and small values of the shear rate, the model predicts the appearance of a sort of strain rate softening, i.e., a reduction of the stress ratio when increasing the shear rate, at low values of the void index. This result, as well as other assumptions made in building the model (e.g., the fact that both the collisional and the quasi-static normal stress diverge at the same concentration), although physically sound, require to be tested against physical experiments and/or numerical simulations.

5 List of symbols

a	material coefficient	[-]
\mathcal{A}	auxiliary function	[-]
\mathcal{B}	auxiliary function	[-]
c	material coefficient	[-]
d	particle diameter	[m]
e	void index	[-]
E	Young's modulus	[Pa]
f_i	$i = 0$ to 5, auxiliary functions	[-]
\mathcal{F}	yield locus	[-]
F	auxiliary function	[-]
g_0	radial distribution function	[-]
G	auxiliary function	[-]
J	auxiliary function	[-]
K	particle stiffness	[Pa m]
L	correlation length	[m]
s	mean separation distance	[m]
t_m	microscopic time scale of particle rearrangement	[s]
T	granular temperature	[m ² /s ²]
$\dot{\gamma}$	shear rate	[1/s]
$\tilde{\gamma}$	fluidity parameter	[1/s]
γ_1, γ_2	auxiliary functions	[-]
Γ	rate of dissipation of fluctuating energy	[Pa/s]
ϵ	effective coefficient of collisional restitution	[-]
ν	concentration	[-]
ν^*	minimum concentration for having strain rate softening when quasi-static stresses are present	[-]
ν_m	maximum concentration in steady, simple shear flows	[-]
ν_{rcp}	concentration at random close packing	[-]
ν_{rlp}	concentration at random loose packing	[-]
ν_s	concentration at which the shear rigidity develops	[-]
ρ_p	particle density	[kg/m ³]
σ	total normal stress	[Pa]

σ_c	collisional normal stress	[Pa]
σ_q	quasi-static normal stress	[Pa]
τ	total shear stress	[Pa]
τ_c	collisional shear stress	[Pa]
τ_q	quasi-static shear stress	[Pa]
ϕ_c	critical friction angle in simple shear conditions	[-]
μ	inter-particle friction coefficient	[-]
Φ	viscous nucleus	[-]

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